MATH 732: CUBIC HYPERSURFACES

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1. Hodge Numbers and Twisted Hodge Numbers

These notes are based on [Huy23, §1.2] and [Huy23, §1.4]. Please see the disclaimer section.

Given a complex manifold X, the *Poincaré lemma* implies that the deRham complex:

$$\Omega_X^* := [0 \to \mathcal{O} \to \Omega_X \to \Omega_X^1 \to \cdots] \qquad \text{(de Rham Complex)}$$

is quasi-isomorphic to the constant sheaf $\mathbf{C}_X$. So the sheaf cohomology of $\mathbf{C}_X$ can be computed by taking the hyper-cohomology of this complex. This complex has a natural *truncation filtration*

$$F^p\Omega_X := [\cdots \to 0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots].$$

If X is a Kähler manifold (e.g. if X is a projective variety) then the spectral sequence associated to this filtered complex "degenerates at the E_1 page." Which means that there is a direct sum decomposition:

$$\bigoplus_{p+q=n} \mathrm{H}^p(X, \Omega^q) = \mathrm{H}^n(X, \mathbf{C}) = \mathrm{H}^n(X, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}. \quad \text{(Hodge decomposition)}$$

This decomposition is not connected to any decomposition of $\mathrm{H}^n(X, \mathbf{R})$, but the group $\mathrm{H}^{p,q}(X) = \mathrm{H}^q(X, \Omega^p)$ is complex conjugate to the group $\mathrm{H}^{q,p}(X)$. The (p,q) *Hodge number* of X is the number

$$h^{p,q}(X) = \dim_{\mathbf{C}} \mathrm{H}^{p,q}(X).$$

These groups are functorial. The Hodge numbers are sometimes collected into the *Hodge diamond* of X.

Example 1.1. When $X \subseteq \mathbf{CP}^2$, is a smooth degree 4 curve, the Hodge diamond is

$$\begin{array}{ccc} & h^{1,1} = 1 & \\ h^{1,0} = 3 & & h^{0,1} = 3 \\ & h^{0,0} = 1 & \end{array}$$

Example 1.2. The Hodge diamond of a quartic (K3) surface is:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The Hodge diamond has several symmetries coming from Poincaré duality, Serre duality, and the Hard Lefschetz theorem.

Just like for cohomology, given a hypersurface $X \subseteq \mathbf{P}^{n+1}$ over an algebraically closed field, the Hodge numbers

$$h^{p,q}(X) = \dim \mathrm{H}^q(X, \Omega^p_X)$$

are again determined by projective space.

To start, we recall the so called *Bott Vanishing* result for $\mathbf{P} = \mathbf{P}(V) = \mathbf{P}^{n+1}$ which can be deduced from the Euler sequence

$$0 \to \Omega_{\mathbf{P}} \to V^\vee \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0. \qquad \text{(Euler Sequence)}$$

Taking exterior powers induces a filtration:

$$0 \to \Omega^p_{\mathbf{P}} \to \wedge^p(V^\vee \otimes \mathcal{O}(-1)) \to \Omega^{p-1}_{\mathbf{P}} \to 0.$$

Exercise 1. *Prove that if X is a variety and*

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{L} \to 0$$

is a short exact sequence of vector bundles (with $\mathcal{L}$ a line bundle), then for any $p > 0$

$$0 \to \wedge^p \mathcal{E}_1 \to \wedge^p \mathcal{E}_2 \to (\wedge^{p-1}\mathcal{E}_1) \otimes \mathcal{L} \to 0.$$

Theorem 1.3 (Bott Vanishing)**.** *Unless (p,q,k) are in the following list of exceptions, we have $\mathrm{H}^q(\mathbf{P}, \Omega^p_X(k)) = 0$.*

List of exceptions:

(1) *If $0 \le p = q \le n$ and $k = 0$, then $\dim \mathrm{H}^p(\mathbf{P}, \Omega^p_{\mathbf{P}}) = 1$.*
(2) *If $q = 0$ and $k > p$ then $\dim \mathrm{H}^0(X, \Omega^p_{\mathbf{P}}(k)) = \binom{n+1+k-p}{k} \cdot \binom{k-1}{p}$.*
(3) *If $q = n+1$, $k < p-(n+1)$, then $\dim H^{n+1}(\mathbf{P}, \Omega^p(k)) = \binom{-k+p}{-k} \cdot \binom{-k-1}{n+1-p}$*

Proof. Let's look at the case $p = 1$, $q < n+1$. Twisting the Euler sequence by k we have:

$$0 \to \Omega_{\mathbf{P}}(k) \to V^\vee \otimes \mathcal{O}(k-1) \to \mathcal{O}(k) \to 0. \qquad \text{(Euler Sequence)}$$

The cohomology of line bundles on $\mathbf{P}$ is given by:

$$\dim \mathrm{H}^q(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k)) \simeq \begin{cases} 0 & 0 < q < n+1 \\ \binom{k+n+1}{k} & q = 0 \\ \binom{k-1}{n+1} & q = n+1 \end{cases}$$

Consequentually,

$$\dim \mathrm{H}^q(\mathbf{P}, V^\vee \otimes \mathcal{O}_{\mathbf{P}}(k-1)) \simeq \begin{cases} 0 & 0 < q < n+1 \\ \binom{k+n}{k-1} \cdot (n+2) & q = 0 \\ \binom{k-2}{n+1} \cdot (n+2) & q = n+1 \end{cases}$$

Not worrying about the case $n = 0$, there are 2 sequences to consider:

$$0 \to \mathrm{H}^0(\Omega_{\mathbf{P}}(k)) \to \mathrm{H}^0(V^\vee(k-1)) \to \mathrm{H}^0(\mathcal{O}_{\mathbf{P}}(k)) \to \mathrm{H}^1(\Omega_{\mathbf{P}}(k)) \to 0.$$

$$0 \to \mathrm{H}^{n+1}(\Omega_{\mathbf{P}}(k)) \to \mathrm{H}^{n+1}(V^\vee(k-1)) \to \mathrm{H}^{n+1}(\mathcal{O}_{\mathbf{P}}(k)) \to 0.$$

A "direct calculation" shows that the middle map on the first sequence is surjective except when $k = 0$. In this case we see that $\mathrm{H}^1(\Omega_{\mathbf{P}})$ is one-dimensional. When $k < 0$: $\mathrm{H}^0(\Omega_{\mathbf{P}}(k)) = \mathrm{H}^1(\Omega_{\mathbf{P}}(k)) = 0$. When $k > 0$ we have:

$$\dim H^0(\Omega_{\mathbf{P}}(k)) = \binom{k+n}{k-1} \cdot (n+2) - \binom{k+n+1}{k}$$

This should equal $\binom{n+k}{k} \cdot \binom{k-1}{1}$ by elementary combinatorics (?). □

Remark 1.4. The slogan is, there is no intermediate cohomology except topological cohomology. All the global sections come from the Euler sequence, and all the top cohomology comes from Serre duality.

Exercise 2. *Assuming the cohomology of line bundles on* $\mathbf{P}$*, prove the theorem using the Euler sequence and its exterior powers.*

We return to the situation of a smooth hypersurface $X \subseteq \mathbf{P}$. There are three relevant sequences:

$$0 \to \Omega^p_{\mathbf{P}}(-d) \to \Omega^p_{\mathbf{P}} \to \Omega^p_{\mathbf{P}}|_X \to 0, \qquad \text{(restriction to } X\text{)}$$

$$0 \to \mathcal{O}_X(-d) \to \Omega_{\mathbf{P}}|_X \to \Omega_X \to 0, \qquad \text{(conormal sequence)}$$

$$0 \to (\wedge^{p-1}\Omega_X)(-d) \to \wedge^p \Omega_{\mathbf{P}}|_X \to \wedge^p \Omega_X \to 0. \qquad (\wedge^p \text{ of conormal})$$

Exercise 3. *Compute the canonical bundle of* X*:* $\omega_X := \wedge^n \Omega_X$*.*

Corollary 1.5. *For* $k < d$*, the natural map:*

$$\mathrm{H}^q(\mathbf{P}, \Omega^p_{\mathbf{P}}(k)) \to \mathrm{H}^q(X, \Omega^p_X(k))$$

is bijective for $p + q < n$ *and injective for* $p + q \le n$*.*

Proof. This is a direct consequence of Bott vanishing for $\mathbf{P}$. □

Remark 1.6. In particular this shows that *Kodaira vanishing* holds for smooth hypersurfaces over any field. I.e., when $k > 0$ and $p + q > n$:

$$\mathrm{H}^q(X, \Omega^p_X(k)) = 0. \qquad \text{(Kodaira vanishing)}$$

These sequences let us compute the Hodge diamond for hypersurfaces in any explicit case. In particular, the corollary (and Serre duality) says that if $p + q \neq n$ then:

$$h^{p,q}(X) = \begin{cases} 1 & 0 \leq p = q \leq n, p + q \neq n \\ 0 & p \neq q \text{ and } p + q \neq n. \end{cases}$$

In other word, understanding the Hodge diamond boils down to understanding the case $p + q = n$.

Example 1.7. Let's compute a few examples low-dimensional examples for cubics. Note, that when X is a cubic, $\omega_X = \mathcal{O}_X(3 - n - 2)$. Thus $h^{n,0}(X) = h^{0,n}(X) = 0$ except when $n = 1$.

$n = 1$ Hodge diamond:

$$\begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix}$$

$n = 2$ Hodge diamond:

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

The number 7 comes out of the following exact sequence:

$$0 \to \mathrm{H}^1(\Omega_{\mathbf{P}}|_X) \to \mathrm{H}^1(\Omega_X) \to \mathrm{H}^2(\mathcal{O}_X(-3)) \to \mathrm{H}^2(\Omega_{\mathbf{P}}|_X) \to 0.$$

Now $h^1(\Omega_{\mathbf{P}}|_X) = 1$, $h^2(\mathcal{O}_X(-3)) = 10$ and $h^2(\Omega_{\mathbf{P}}|_X) = 4$, which gives the answer 7.

$n = 3$ Hodge diamond:

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 10 & & 10 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$

$n = 4$ Hodge diamond:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

$n = 5$ Hodge diamond:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
0 & & 0 & & 21 & & 21 & & 0 & & 0 \\
 & 0 & & 0 & & 1 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 1 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

Remark 1.8. The *primitive* Hodge numbers fit into a generating series (Hirzebruch?). That is, if $X_n \subseteq \mathbf{CP}^{n+1}$ is a smooth degree d, n dimensional hypersurface, then

$$\sum_{p,q\geq 0, n\geq 0} h^{p,q}(X)_{\mathrm{pr}} y^p z^q = \frac{1}{(1+y)(1+z)} \cdot \left[\frac{(1+y)^d - (1+z)^d}{(1+z)^d y + (1+y)^d z} - 1 \right].$$

Exercise 4. *Show that the cotangent bundle Ω_X of a smooth hypersurface of degree $d \geq 3$ is stable.*

References

[Huy23] Daniel Huybrechts. *The geometry of cubic hypersurfaces*, volume 206 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023.